Slipped non-Positive Reduced Dynamics and Entanglement

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Non-positive Markov approximations are sometimes used to describe the dynamics of qubits in weak interaction with suitable environments; the appearance of negative probabilities is avoided by assuming that the transient regime eliminates from the possible initial conditions those qubit states which would otherwise be mapped out of the Bloch sphere by the subsequent Markovian time-evolution. By means of a simple model, we discuss some physical inconsistencies of this approach in relation to entanglement; in particular, we show that slipped non-positive reduced dynamics might create entanglement through a purely local action.

I. INTRODUCTION

Semigroups of dynamical maps are used to describe the time-evolution of open quantum systems S in weak interaction with suitable external environments, typically an infinite heath bath in equilibrium at a given temperature acting as a source of dissipation and noise. They have been successfully used in many phenomenological applications in quantum chemistry, quantum optics, statistical physics [1, 2, 3, 4, 5].

In the following, we shall consider qubits described by density matrices ρ, corresponding to three dimensional real vectors of length ≤ 1 in the Bloch sphere, that evolve in time under the action of semigroups of linear maps γ_t, $t \geq 0$. Formally, these arise from the exponentiation of a generator $\mathbb{L}$, $\gamma_t = \exp(t\mathbb{L})$, and satisfy the forward in time composition law $\gamma_t \circ \gamma_s = \gamma_{t+s}$, $t, s \geq 0$.

A preliminary natural request on the maps γ_t is that they send any initial state ρ into another state $\gamma_t[\rho]$ at all $t \geq 0$, namely that they map the Bloch sphere into itself. Only in such a way, the spectrum of an evolving ρ remains positive and its eigenvalues can be interpreted as probabilities.

Such a property of the maps γ_t is called *positivity*. In line of principle, it is not sufficient to guarantee full physical consistency of the maps γ_t: a more restrictive property, namely *complete positivity*, need be imposed [6, 7, 8]. In such a way, not only the positivity of γ_t is guaranteed, but also that of the amplified map $\gamma_t \otimes \mathrm{id}$; this describes the time-evolution of a qubit S statistically coupled to an ancillary qubit which remains inert under the action of the identity operation id. Moreover, complete positivity of γ_t fully characterizes the form of the generator $\mathbb{L}$ [9, 10]. In particular, in the case of a qubit system, there appears a characteristic order relation ($2T_1 \geq T_2$) between the decay times of the diagonal (T_1) and off-diagonal (T_2) entries of its time-evolving density matrix [9].

The semigroup γ_t describes the reduced dynamics of the immersed qubit S when the environment degrees of freedom have been eliminated and the memory effects due to a short transient regime have been got rid of by means of suitable Markovian approximations. If not performed with due care, these latter lead to reduced dynamics neither completely positive, nor even positive.

Absence of positivity of γ_t means that there are initial density matrices ρ that may, in the course of time, develop negative eigenvalues and thus lose their meaning as physical states. If one wants to stick to non-positive reduced dynamics, a possible way out of physical inconsistencies amounts to assuming that not all density matrices are allowed as initial conditions for γ_t, but only those which do not develop negative eigenvalues. The mechanism which eliminates the unwanted initial states is ascribed to the transient regime which rules the time behavior of the subsystem S before one can legitimately use the semigroup γ_t. Namely, prior to γ_t, the environment action on the subsystem is via a map $\mathbb{S}$ that projects the whole state space of S, $\mathcal{S}(S)$, into a subset of "good" states $\mathbb{S}(\mathcal{S}(S))$, so that $\gamma_t \circ \mathbb{S}$ acts as a positive map on $\mathcal{S}(S)$, even if γ_t does not.

The map $\mathbb{S}$ is known in the literature as *slippage of initial conditions* [11, 12, 13, 14, 15], its introduction being motivated by the the difficulty to accept that a physical effect like the decay-times hierarchy of a qubit be induced by its possible entanglement with an uncontrollable external ancillary qubit, a seemingly academic, abstract scenario with philosophical overtones.

On the other hand, quantum information and communication theories have amply demonstrated the role of entanglement as a concrete physical resource and developed techniques both theoretical and experimental for its manipulation [16]. It is thus not only of academical interest to study the slippage-approach in relation to entanglement; more precisely, we shall be interested in setting the ground to answer the following question. Suppose the inconsistencies of a non-positive γ_t have been cured by considering a *slipped* $\gamma_t \circ \mathbb{S}$. Which kind of effects has $(\gamma_t \circ \mathbb{S}) \otimes \mathrm{id}$ on the entangled states of $S + S$?

Aim of this paper is to extend a previous result [17] in order to indicate which kind of pathologies may arise from amplifications of slipped non-positive semigroups acting on bipartite systems. Given an environment, its associated slippage operator $\mathbb{S}$ is far from being technically accessible as a mathematical object and such are its effective properties. We shall thus model $\mathbb{S}$ by the simplest map with the prescribed slippage properties and then discuss its action on the isotropic states [18] of a two qubit systems, one qubit immersed in a dephasing environment and the other one external to it. We shall show that, by preceding it by a suitable slippage operator $\mathbb{S}$, a non-positive reduced dynamical map γ_t can indeed be turned into a completely positive map $\gamma_t \circ \mathbb{S}$, but also that, unless strongly slipped, the maps γ_t can, acting locally, create entanglement, a fundamentally non-local property. Furthermore, we shall present instances of cases where the necessary strong slippage results in the elimination of all isotropic entangled states. We take this as an indication of a general behavior: in the weak-coupling regime, slipped non-completely positive dynamics appear to be incompatible with entanglement.

II. QUANTUM DYNAMICAL SEMIGROUPS

In this section, we shall briefly review the standard approach to open quantum dynamics, namely the so-called weak-coupling limit.

The typical physical context is represented by a finite-dimensional (n-level) subsystem S in weak interaction with an environment E, this latter corresponding to an infinite dimensional reservoir in equilibrium at some fixed temperature T. The space of states $\mathcal{S}(S)$ of S consists of $n \times n$ density matrices ρ_S, while the state of the environment ρ_E is taken to be an equilibrium state ρ_E. The dynamics of the (closed) system $S + E$ is typically described by a Hamiltonian of the form

$$H_{S+E} = H_S + H_E + g\, H_I \ , \quad H_I = \sum_\alpha X_S^\alpha \otimes X_E^\alpha \ , \tag{1}$$

where H_S is the subsystem Hamiltonian, H_I an interaction term, of strength g, linear in the system and (centered) environment operators $X^\alpha_{S,E}$ $(\mathrm{Tr}_E(\rho_E X_E^\alpha) = 0)$.

When the interaction between S and E is weak ($g << 1$), it makes sense to derive a so-called *reduced dynamics*, namely a description of the time-evolution of S involving the system S alone. The derivation amounts to the elimination of the degrees of freedom of E while retaining the effects of their presence on S; the resulting time-evolution is irreversible, in general non-linear, and dominated by memory effects. By assuming the initial state of $S+E$ to be of the uncorrelated form $\rho_S \otimes \rho_E$, non-linearities are automatically eliminated and the standard *projection technique* leads to the following equation of motion for $\rho_S \in \mathcal{S}(S)$ [2]:

$$\partial_t \rho_S(t) = -i\Big[H_S\, ,\, \rho_S(t)\Big] + g^2 \int_0^t \mathrm{d}s\, \mathbb{K}(s)[\rho_S(t-s)] \ , \tag{2}$$

where $\mathbb{K}(s)$ is a highly complicated kernel that, acting on the states of S, takes into account both the degrees of freedom of E and the time-evolution of all of them prior to time t.

The ensuing dynamics preserves the trace of ρ_S and is completely positive in the sense that it sends any initial ρ_S into a $\rho_S(t)$ which results from the action of a linear map $\mathbb{G}_t$:

$$\rho \mapsto \rho_S(t) =: \mathbb{G}_t[\rho_S] = \sum_j V_j(t)\, \rho_S\, V_j^\dagger(t) \ , \tag{3}$$

where the $V_j(t)$ are $n \times n$ matrices such that $\sum_j V_j^\dagger(t) V_j(t) = 1$.

Remarks 2.1

1. A trace-preserving linear map $\mathbb{G}$ on $\mathcal{S}(S)$ is called *positive* if $\mathbb{G}[\rho]$ has positive spectrum for any $\rho \in \mathcal{S}(S)$. If $\mathbb{G}$ is *completely positive*, then it is not only positive, but, upon coupling S with another n-level system system, the amplified map $\mathbb{G}\otimes\mathrm{id}$ preserves the positivity of all possible states of the compound system $S+S$, where id means that the ancillary system S is not affected. It turns out that complete positivity is equivalent to $\mathbb{G}$ being of the Kraus-Stinespring form [7, 8]

$$\mathbb{G}[\rho] = \sum_j G_j\, \rho\, G_j^\dagger \ , \quad \sum_j G_j^\dagger G_j = 1 \ . \tag{4}$$

2. The physical meaning of complete positivity is intimately related to the phenomenon of quantum entanglement [5]; therefore, its physical role can be appreciated only considering correlated bipartite quantum systems of which only one party undergoes a state change. In fact, if coupling of S to ancillas could be excluded, the physical transformations of the states of S might adequately be described by positive linear maps $\mathbb{G}$. Also, if S is coupled to another S, amplified positive maps $\mathbb{G}\otimes\mathrm{id}$ would preserve the positivity of any separable state
$$\rho^{sep}_{S+S} := \sum_{ij} \lambda_{ij} \rho^i_S \otimes \rho^j_S \ , \qquad \rho^{i,j}_S \in \mathcal{S}(S) \ . \tag{5}$$
However, since there are states of $S+S$, the *entangled* ones, which cannot be written as in (5), when $\mathbb{G}\otimes\mathrm{id}$ acts on anyone of them, the positivity of the eigenvalues of $\mathbb{G}\otimes\mathrm{id}[\rho^{ent}_{S+S}]$ can be preserved if and only if $\mathbb{G}$ is not only positive, but also completely positive.

3. If a linear map $\mathbb{G}$ on the states of S is not completely positive, this does not mean that $\mathbb{G}\otimes\mathrm{id}$ sends any entangled state of $S+S$ out of the class of states: some entangled states may none the less have the positivity of their spectrum preserved at all times. However, for any non-completely positive $\mathbb{G}$, the totally symmetric projection
$$P := \frac{1}{n}\sum_{i,j=1}^{n} |i\rangle\langle j| \otimes |i\rangle\langle j| \ , \tag{6}$$
will always have some negative eigenvalues appearing in the spectrum of $\gamma_t\otimes\mathrm{id}[P]$. Indeed, a theorem of Jamolkiowski [18] establishes that $\mathbb{G}\otimes\mathrm{id}[P]$ is a positive matrix if and only if $\mathbb{G}$ is completely positive.

Beside being practically impossible to handle mathematically, the dynamical maps $\mathbb{G}_t$ do not satisfy a forward-in-time composition law: $\mathbb{G}_t\circ\mathbb{G}_s \neq \mathbb{G}_{t+s}$. Manageable Markovian approximations are obtained by weak-coupling limit techniques [2]; these exploit the weakness of the interaction between S and E. Indeed, (2) implies that the influence of the environment becomes visible on a time scale such that $g^2 t = \tau$; by setting $t = \tau/g^2$ in (2), the typical prescription is to let $g \to 0$ and approximate $\int_0^{\tau/g^2}$ by $\int_0^\infty$ and $\rho(t-s) = \rho(\tau/g^2 - s)$ by $\rho(\tau/g^2) = \rho(t)$. One thus obtains the equations
$$\partial_t \rho_S(t) = -i\left[\widetilde{H}_S \, , \, \rho_S(t)\right] + g^2\,\mathbb{D}[\rho_S(t)] \ , \tag{7}$$

where $\mathbb{D} := \int_0^\infty \mathrm{d}s\, \mathbb{K}(s)$ and $\widetilde{H}_S$ is a Lamb-shifted subsystem Hamiltonian corrected by terms of order g^2.

The linear map $\mathbb{D}$ takes into account the environment induced dissipative effects, essentially damping and noise, affecting the subsystem S when it weakly interacts with its environment E. By introducing a Hilbert-Schmidt orthonormal set of $n \times n$ matrices F_i, $i = 1, 2, \ldots, n^2 - 1$, $F_{n^2} = 1/\sqrt{n}$, $\mathrm{Tr}(F_i^\dagger F_j) = \delta_{ij}$, $\mathbb{D}$ can always be put in the form

$$\mathbb{D}[\rho_S] = \sum_{i,j=1}^{n^2-1} C_{ij} \Big(F_i\, \rho_S\, F_j^\dagger \, - \frac{1}{2}\Big\{ F_j^\dagger F_i \,,\, \rho \Big\}\Big) \ , \tag{8}$$

where $C = [C_{ij}]$ is a self-adjoint $(n^2 - 1) \times (n^2 - 1)$ coefficient matrix, the so-called *Kossakowski matrix* [9].

By setting the right hand side of (8) equal to $\mathbb{L}[\rho_S(t)]$, the resulting Markovian reduced dynamics consists of a semigroup of linear maps $\gamma_t = \exp(t\mathbb{L})$; these maps are trace-preserving and their positivity or complete positivity depends on the matrix C:

1. if the Kossakowski matrix $C \geq 0$, then the maps γ_t are completely positive and vice versa [22];

2. if C is not positive, no general necessary and sufficient conditions are known which yield positive γ_t (see [5] for certain cases where this possibility indeed exists).

III. AN OPEN QUBIT SYSTEM

We now follow and develop an argument developed in [17] and consider, as a concrete model, a single qubit, represented by a spin 1/2 particle coupled to a constant magnetic field vertically directed and to a weak stochastic classical magnetic field $\mathbf{G} = (G_1(t), G_2(t), G_3(t))$. The time-evolution of the qubit density matrices is determined by the time-dependent Liouville-von Neumann equation

$$\partial_t \rho_S(t) = -i\Big[\widetilde{\omega}\, \sigma_3 \,,\, \rho_S(t)\Big] \, - i\left[\sum_{i=1}^{3} G_i(t)\, \sigma_i \,,\, \rho_S(t)\right] \ , \tag{9}$$

where the σ_i are the Pauli matrices and the scalar quantities $G_i(t)$ are chosen to be stationary, stochastic variables with vanishing first moments and diagonal covariance matrix:

$$\langle G_i(t) G_j(s) \rangle = G_i \mathrm{e}^{-\lambda_i (t-s)} \delta_{ij} \ , \tag{10}$$

with $G_1 > G_2 > 0$, $G_3 > 0$ and $\lambda_1 = \lambda_2 = \lambda > 0$. Such a classical stochastic field is a convenient modeling of a heat bath whose temperature is sufficiently high, whereby the global Hamiltonian (1) for the system plus environment may be replaced by (9).

Of course, the solution of the above stochastic equation is a stochastic density matrix; useful physical information may only come by considering the average of $\rho_S(t)$ with respect to the noise. Techniques based on the weak-coupling limit can be applied to (9) [19]; in general, these lead to a reduced dynamics which suffers from lack of positivity of the ensuing Markovian reduced dynamics.

With reference to the evolution equation (7) with the dissipative term as in (8), choosing $F_i = \dfrac{\sigma_i}{\sqrt{2}}$, $i = 1, 2, 3$, and $F_4 = \dfrac{1}{\sqrt{2}}$, one gets

$$\widetilde{H} = \omega\sigma_3 \ , \quad \omega := \widetilde{\omega}\left(1 + \frac{2(G_1 + G_2)}{\lambda^2 + 4\widetilde{\omega}^2}\right) \tag{11}$$

$$C = \begin{pmatrix} \alpha_1 & -b & 0 \\ -b & \alpha_2 & 0 \\ 0 & 0 & a \end{pmatrix} \ , \quad \alpha_{1,2} := \frac{2G_{1,2}\lambda}{\lambda^2 + 4\,\widetilde{\omega}^2} \ , \ a = \frac{2G_3}{\lambda_3} \ , \ b := 2\widetilde{\omega}\frac{G_2 - G_1}{\lambda^2 + 4\,\widetilde{\omega}^2} \ . \tag{12}$$

In order to better expose the consequences of sticking to non-positive dissipative semigroups, we simplify the analysis by choosing $\widetilde{\omega} \ll \lambda_3 \ll \lambda$ and $\lambda\, G_{1,2} \ll \widetilde{\omega}\,|G_2 - G_1|$. This allows us to neglect $\alpha_{1,2}$ with respect to both b and a; we shall thus deal with the dissipative evolution equation

$$\partial_t\rho_S(t) = -i\big[\omega\sigma_3\,,\,\rho_S(t)\big] + a\Big(\sigma_3\,\rho_S(t)\,\sigma_3 - \rho_S(t)\Big) - b\,\Big(\sigma_1\,\rho_S(t)\,\sigma_2 + \sigma_2\,\rho_S(t)\,\sigma_1\Big) \ . \tag{13}$$

In order to solve the above equation and study the ensuing time-evolution, it proves convenient to recast the qubit density matrices as

$$\rho = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{12}^* & 1 - \rho_{11} \end{pmatrix} = \frac{1}{2}(1 + \mathbf{r}\cdot\sigma) \ , \tag{14}$$

where $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ and $\mathbf{r} = (r_1, r_2, r_3)$, with

$$r_1 = 2\Re(\rho_{12}) \ , \ r_2 = -2\Im(\rho_{12}) \ , \ r_3 = 2\rho_{11} - 1 \ . \tag{15}$$

The matrix ρ is positive if and only if its determinant

$$\mathrm{Det}(\rho) = \frac{1 - \|\mathbf{r}\|^2}{4} \geq 0 \ , \tag{16}$$

whence each state is identified by a vector $\mathbf{r} \in \mathbb{R}^3$ of norm $\|\mathbf{r}\| \leq 1$, that is by a point in the so called Bloch sphere.

If the qubit is weakly interacting with an environment, its dissipative dynamics should be described by a semigroup of trace-preserving maps γ_t sending any state ρ at $t = 0$ into a state $\gamma_t[\rho]$ at time $t > 0$. According to (16), this is equivalent to $|\mathbf{r}_t| \leq 1$, where $\mathbf{r}_t = (r_1(t), r_2(t), r_3(t)) \in \mathbb{R}^3$ is the Bloch vector identifying

$$\gamma_t[\rho] = \frac{1}{2}\Big(1 + \sum_{i=1}^{3} r_i \gamma_t[\sigma_i]\Big) = \frac{1}{2}\Big(1 + \sum_{i=1}^{3} r_i(t)\sigma_i\Big) \ . \tag{17}$$

In particular, (13) translates into a linear differential equation for the Bloch vector

$$\begin{pmatrix} \dot{r}_1 \\ \dot{r}_2 \\ \dot{r}_3 \end{pmatrix} = -2\mathcal{L} \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} \ , \tag{18}$$

where the 3×3 matrix $\mathcal{L} = \mathcal{H} + \mathcal{D}$ consists of the sum of an antisymmetric component $\mathcal{H}$ corresponding to the commutator with the Hamiltonian in (7), while $\mathcal{D}$ is symmetric and corresponds to the dissipative term $\mathbb{D}$. Explicitly,

$$\mathcal{L} = \begin{pmatrix} a & b+\omega & 0 \\ b-\omega & a & 0 \\ 0 & 0 & 0 \end{pmatrix} \ , \ \mathcal{H} = \begin{pmatrix} 0 & \omega & 0 \\ -\omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \ , \ \mathcal{D} = \begin{pmatrix} a & b & 0 \\ b & a & 0 \\ 0 & 0 & 0 \end{pmatrix} \ . \tag{19}$$

The semigroup of maps $\gamma_t = \mathrm{e}^{t\mathbb{L}}$ on the state-space $\mathcal{S}(S)$ generated by (13) corresponds to a semigroup of 3×3 matrices $\mathcal{G}_t = \mathrm{e}^{-2t\mathcal{L}}$ acting on the Bloch sphere.

From (18), (19) and (13) it follows that

$$\begin{cases} \dot{r}_1 = -2a\, r_1 - 2(b+\omega)\, r_2 \\ \dot{r}_2 = -2(b-\omega)\, r_1 - 2a\, r_2 \\ \dot{r}_3 = 0 \end{cases} \ . \tag{20}$$

Setting $\Omega^2 = \omega^2 - b^2$ one gets:

$$\begin{cases} r_1(t) = \mathrm{e}^{-2at}\Big[r_1 \cos(2\Omega t) - r_2 \frac{\omega+b}{\Omega} \sin(2\Omega t)\Big] \\ r_2(t) = \mathrm{e}^{-2at}\Big[r_1 \frac{\omega-b}{\Omega} \sin(2\Omega t) + r_2 \cos(2\Omega t)\Big] \\ r_3(t) = r_3 \end{cases} \quad \text{and} \tag{21}$$

Apparently, the action of the semigroup is that of a dephasing channel with no influence on the diagonal elements of any initial ρ: it remains to be checked whether the matrices $\mathcal{G}_t$ correspond to non-positive, positive or completely positive maps γ_t.

The Kossakowski matrix in (12) is positive, hence γ_t completely positive, if and only if $b = 0$.

Let instead $b \neq 0$ and consider the Bloch vectors $\mathbf{r}_\pm = 1/\sqrt{2}(0, 1, \pm 1, 0)$ corresponding to the pure states $(\mathrm{Det}(\rho_\pm) = 0)$ $\rho_\pm = \frac{1}{2}(1 + \sigma_1 \pm \sigma_2)$. At $t > 0$ they evolve into matrices of trace 1 with Bloch vectors such that

$$\|\mathbf{r}_t^\pm\|^2 = \mathrm{e}^{-4at}\left[\cos^2(2\Omega t) + \frac{\omega^2 + b^2}{\Omega^2}\sin^2(2\Omega t) \mp \frac{b}{\Omega}\sin(4\Omega t)\right] \simeq 1 - 4t(a \pm b) \ , \qquad (22)$$

for $t \to 0$. It thus follows that γ_t acts in a physically consistent way on the states of the qubit only if $a^2 - b^2 \geq 0$. Actually, this condition is also sufficient for γ_t to preserve the positivity of states for all $t > 0$. Indeed, $a^2 - b^2 \geq 0$ corresponds to the positivity of the matrix $\mathcal{D}$ in (19). Now, a state ρ exits the Bloch sphere at time t if and only if the Bloch vector of $\gamma_t[\rho]$ is such that $\mathbf{r}_t = 1$ and

$$\frac{\mathrm{d}\|\mathbf{r}_t\|^2}{\mathrm{d}t} = -2\langle \mathbf{r}_t|\mathcal{D}|\mathbf{r}_t\rangle = -2a(r_1^2(t) + r_2^2(t)) - 2br_1(t)r_2(t) > 0 \ . \qquad (23)$$

Thus, when $\mathcal{D} \geq 0$, the state moves from the surface towards the interior of the Bloch sphere if $\langle \mathbf{r}_t|\mathcal{D}|\mathbf{r}_t\rangle > 0$, while if $\langle \mathbf{r}_t|\mathcal{D}|\mathbf{r}_t\rangle = 0$, then it remains of norm 1.

Proposition 3.1 The semigroup $\mathcal{G}_t = \mathrm{e}^{-2t\mathcal{L}}$ generated by (18) with $\mathcal{L}$ as in (19) corresponds to a semigroup of linear maps γ_t on the state-space of the qubit S which are

1. completely positive if and only if $b = 0$;
2. only positive if and only if $b \neq 0$ and $a^2 - b^2 \geq 0$;
3. not even positive if $0 \leq a < b$.

IV. SLIPPAGE OF INITIAL CONDITIONS

From the considerations in Remark 2.1.2, in order to avoid physical inconsistencies that may arise from non-completely positive reduced dynamics in presence of entangled states, it would be rather natural to consider only reduced dynamics enjoying complete positivity and to exclude those that do not. Actually, the problem is more acute since those Markovian approximations γ_t which fail to be completely positive are sometimes not even positive. This means that their pathological behaviour, of the kind discussed in the previous section, already appears with a single qubit.

At the level of a single system, if one wants to stick to non-positive reduced dynamics as the best Markovian approximations to the actual dissipative dynamics, a possible way out of the physical inconsistencies is known in the literaure as *slippage of initial conditions* [11, 12, 13, 14, 15].

The argument goes roughly as follows. Not all density matrices ρ_S evolve into $\gamma_t[\rho_S]$ with negative eigenvalues in their spectra; thus, let $\mathcal{S}^*(S) \subseteq \mathcal{S}(S)$ denote the subset of states which remain positive under γ_t for all $t > 0$ and let $\mathbb{S} : \mathcal{S}(S) \mapsto \mathcal{S}^*(S)$ be a map selecting this subset of "good states" for γ_t. Then, even though the maps γ_t suffer from physical inconsistencies, the maps $\gamma_t \circ \mathbb{S}$ do not.

The slippage of initial conditions is supposed to be operated by the transient regime prior to the Markov one. In other words, according to (2),

$$\mathbb{S}[\rho_S] \simeq g^2 \int_0^{t_{trans}} \mathrm{d}s\, \mathbb{K}(s)[\rho_S(t_{trans} - s)] \ , \tag{24}$$

where t_{trans} is the span of time during which memory effects cannot be disregarded. Notice that the weak-coupling limit $g^2 t = \tau$, $g \to 0$, does exactly push $t_{trans} \to 0$. Roughly speaking, the action of the transient regime is supposed to be such that the Markovian dynamics following it does not act on any possible initial density matrix, but only on those selected by $\mathbb{S}$ which are free from the inconsistencies arising from non-positivity.

Of course the plausibility of such an approach has to be tested against actual physical open quantum contexts; however, the highly complex nature of the kernel $\mathbb{K}(t)$ describing the transient, memory-full dynamics, scarcely allows any hope to go beyond extremely rude approximations in describing mathematically what its effects really are. In the following, relative to the reduced dynamics in (21), we shall model a slippage operator in a simple way that however embodies some of its salient features.

We first observe that according to the slippage philosophy, there would be no need to invoke a selection map $\mathbb{S}$ when $a^2 \geq b^2$ for in this case the maps γ_t are positive and cannot drive any initial state outside the Bloch sphere. In other words, all initial 2×2 density matrices are good initial states for the reduced dynamics and one need not resort to a slippage mechanism to select a suitable subclass of them.

We shall then set $a^2 < b^2$; from the arguments of the previous section, the most dangerous states are pure, so that non-pure states within the Bloch sphere are expected to offer less problems. How far into the Bloch sphere should one go in order to be sure to get rid of all those states which γ_t would sooner or later map outside it? It is evident from (21) that Bloch vectors $\mathbf{r}_t$ may exit the Bloch sphere but their lengths $\|\mathbf{r}_t\|$ cannot diverge. The maximum value R achievable by $\|\mathbf{r}_t\|$ can be explicitly obtained.

First, observe that, within and on the Bloch sphere,

$$\|\mathbf{r}_t\|^2 = \langle \mathbf{r}|\mathcal{G}_t^T\mathcal{G}_t|\mathbf{r}\rangle \leq \|\mathcal{G}_t^T\mathcal{G}_t\|\|\mathbf{r}\|^2 \leq \|\mathcal{G}_t^T\mathcal{G}_t\| \ , \tag{25}$$

where $\mathcal{G}_t^T$ is the transposed of $\mathcal{G}_t$ and the operator norm $\|\mathcal{G}_t^T\mathcal{G}_t\|$ equals the largest eigenvalue of the positive, real symmetric matrix

$$\begin{aligned}\mathcal{G}_t^T\mathcal{G}_t \ = \ & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \mathrm{e}^{-4at}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ & + \ \mathrm{e}^{-4at}\frac{2b}{\Omega^2}\sin(2t\Omega)\begin{pmatrix} (b-\omega)\,\sin(2t\Omega) & -\Omega\,\cos(2t\Omega) & 0 \\ -\Omega\,\cos(2t\Omega) & (b+\omega)\,\sin(2t\Omega) & 0 \\ 0 & 0 & 0 \end{pmatrix} \ .\end{aligned} \tag{26}$$

Of the three eigenvalues of the above matrix, one is 1 and the largest of the remaining two is

$$R^2(t) := \mathrm{e}^{-4at}\left(\frac{b}{\Omega}\,|\sin(2t\Omega)| + \sqrt{1 + \frac{b^2}{\Omega^2}\sin^2(2t\Omega)}\right)^2 \ . \tag{27}$$

If $a^2 < b^2$, it can be explicitly computed that $R(t)$ achieves its maximum

$$R := \mathrm{e}^{-2at'}\sqrt{\frac{\omega + \sqrt{b^2 - a^2}}{\omega - \sqrt{b^2 - a^2}}} \qquad \text{at} \quad t' := \frac{1}{2\Omega}\arcsin\left(\frac{\Omega}{b}\sqrt{\frac{b^2 - a^2}{\Omega^2 + a^2}}\right) \ . \tag{28}$$

Remarks 4.1

1. R is surely > 1; indeed, for $a^2 < b^2$, γ_t is not positive, whence the norm of $\mathcal{G}_t^T\mathcal{G}_t$ cannot be ≤ 1.

2. The maximum (28) is achieved at t' starting from the pure state ρ^* whose Bloch vector $\mathbf{r}^*$ is the eigenvector of $\mathcal{G}_t^T\mathcal{G}_t$ relative to the eigenvalue R^2.

Set $0 \leq \mu \leq R^{-1}$ and consider the following linear map on $\mathcal{S}(S)$:

$$\rho \mapsto \mathbb{S}_\mu[\rho] = \frac{1}{2}\big(1 + \mu \mathbf{r} \cdot \sigma\big) \tag{29}$$

Then, because of linearity,

$$\gamma_t \circ \mathbb{S}_\mu[\rho] = \frac{1}{2}\big(1 + \mu \mathbf{r}_t \cdot \sigma\big) \ , \quad \text{and} \quad \|\mu \mathbf{r}_t\| \leq \mu R \leq 1 \ , \tag{30}$$

whence no state can be mapped out of the Bloch sphere by $\gamma_t \circ \mathbb{S}_\mu$ at any $t \geq 0$.

The map $\mathbb{S}_\mu$ is completely positive, its Kraus-Stinespring form being

$$\mathbb{S}_\mu[\rho] = \frac{1+3\mu}{4}\,\rho + \frac{1-\mu}{4}\sum_{i=1}^{3} \sigma_i\,\rho\,\sigma_i \ . \tag{31}$$

Remark 4.2 If the interaction of the subsystem with the environment in equilibrium is weak, then the hypothesis of an initial factorized state $\rho_S \otimes \rho_E$ is not an unphysical restriction, as well as the hypotheisis that the transient act as a linear completely positive operator.

The action of $\mathbb{S}_\mu$ is rather drastic in that it rigidly maps the unit Bloch sphere onto a sphere of smaller radius; one could instead envisage more sophisticated slippage mechanisms. However, the intent of this paper is to show the kind of problems afflicting the simplest possible of them. They should indeed not depend on the explicit model of transient, being it related to the non-positivity of the Markovian approximation and, as we shall presently see, to the existence of entangled states.

V. SLIPPAGE VS ENTANGLEMENT

Thanks to the rapid development of quantum information, the notion of entanglement left the purely epistemological arena in which it was confined for thirty or so years and made its appearance on the foreground of physics as a most precious resource to perform otherwise impossible tasks as dense quantum coding and quantum teleportation [16].

Physically speaking, the model we are going to consider is that of one qubit immersed in a stochastic environment whose effects are described by the maps γ_t of the previous section, which is correlated to another qubit which is a dynamically independent and inert ancilla. This is the physical context when, for instance, an entangled Bell state is constructed in a laboratory and, while one of the two qubit is kept there, the other is sent to a distant party through a noisy channel. The time-evolution of the compound system is thus described by the semigroup of maps $\gamma_t \otimes \mathrm{id}$, where the presence of the bath is felt locally by one of the two qubits only and the other does not evolve at all (this is the meaning of id).

In the following, we shall study how the entanglement content of special qubit states changes under the local time-evolution as above. Concretely, we shall consider the time-evolution of the *concurrence* [21], which, for any state ρ of a two qubit system, is defined by

$$\mathcal{C}(\rho) := \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\} \ , \tag{32}$$

where the λ_i's are the positive square roots of the (positive) eigenvalues of the matrix

$$\rho\,\widetilde{\rho} \ , \quad \widetilde{\rho} := (\sigma_2 \otimes \sigma_2)\,\rho^*\,(\sigma_2 \otimes \sigma_2) \ , \tag{33}$$

where ρ^* is the matrix obtained by taking the complex conjugate of the entries of ρ in a chosen representation.

By means of the techniques sketched above, we shall now study whether curing the non-positivity of reduced dynamics by slipping the initial conditions can nevertheless conflict with the existence of entanglement between the immersed open qubit and its inert ancilla.

Remark 5.1 That problems may easily arise can be argued by observing that in the case of $a^2 > b^2 > 0$, the maps γ_t are positive and, a priori, no slippage is needed. Nevertheless, according to Remark 2.1.1, the totally symmetric projector P is definitely mapped out of the state of space of the compound system $S + S$, for γ_t is completely positive only if $b = 0$. Therefore, if $b \neq 0$, one needs a slippage operator act on the two qubit system; as the latter can only be due to the bath, it must also affect the single qubit immersed in it, independently of whether it is coupled or not to an ancillary qubit.

In order to set the ground for discussing the slippage mechanism in relation to entanglement, we rewrite the totally symmetric projector as follows

$$P = \frac{1}{4}\Big(1 \otimes 1 + \sigma_1 \otimes \sigma_1 - \sigma_2 \otimes \sigma_2 + \sigma_3 \otimes \sigma_3\Big) \ , \tag{34}$$

so that (31) maps it into

$$P_\mu := \mathbb{S}_\mu \circ \mathrm{id}[P] = \frac{1}{4}\Big(1 \otimes 1 + \mu\Big(\sigma_1 \otimes \sigma_1 - \sigma_2 \otimes \sigma_2 + \sigma_3 \otimes \sigma_3\Big)\Big) \ . \tag{35}$$

Then we let $\gamma_t \otimes \mathrm{id}$ act on P_μ and consider $P_\mu(t) := \gamma_t \otimes \mathrm{id}[P_\mu]$; the result is easily computed from

$$\begin{cases} \sigma_1(t) = \mathrm{e}^{-2at}\Big[\sigma_1 \cos(2\Omega t) + \sigma_2 \frac{\omega - b}{\Omega} \sin(2\Omega t)\Big] \\ \sigma_2(t) = \mathrm{e}^{-2at}\Big[-\sigma_1 \frac{\omega + b}{\Omega} \sin(2\Omega t) + \sigma_2 \cos(2\Omega t)\Big] \\ \sigma_3(t) = \sigma_3 \ , \end{cases} \tag{36}$$

which follow using (21) and (17). Explicitly,

$$P_\mu(t) = \frac{1}{4}\begin{pmatrix} 1+\mu & 0 & 0 & 2\mu\, B_t \\ 0 & 1-\mu & 2\mu\, C_t & 0 \\ 0 & 2\mu\, C_t^* & 1-\mu & 0 \\ 2\mu\, B_t^* & 0 & 0 & 1+\mu \end{pmatrix} \ , \ \begin{cases} B_t := \mathrm{e}^{-2at}\Big(\cos(2\Omega t) - i\frac{\omega}{\Omega}\sin(2\Omega t)\Big) \\ \\ C_t := i\, \frac{b}{\Omega}\, \mathrm{e}^{-2at} \sin(2\Omega t) \end{cases} \ . \tag{37}$$

The eigenvalues of this 4×4 matrix are, in decreasing order,

$$e_1^\mu(t) \ := \ \frac{1}{4}\left[1 + \mu\left(1 + 2\mathrm{e}^{-2at}\sqrt{1 + \frac{b^2}{\Omega^2}\sin^2(2\Omega t)}\right)\right] \tag{38}$$

$$e_2^\mu(t) \ := \ \frac{1}{4}\left[1 + \mu\left(1 - 2\mathrm{e}^{-2at}\sqrt{1 + \frac{b^2}{\Omega^2}\sin^2(2\Omega t)}\right)\right] \tag{39}$$

$$e_3^\mu(t) \ := \ \frac{1}{4}\left[1 - \mu\left(1 - 2\mathrm{e}^{-2at}\frac{b}{\Omega}\sin(2\Omega t)\right)\right] \tag{40}$$

$$e_4^\mu(t) \ := \ \frac{1}{4}\left[1 - \mu\left(1 + 2\mathrm{e}^{-2at}\frac{b}{\Omega}\sin(2\Omega t)\right)\right] \ . \tag{41}$$

As already stressed, physical consistency asks that the slippage operator $\mathbb{S}_\mu$ cure not only the non-positivity of γ_t, but also ensure the positivity of the maps $(\gamma_t \circ \mathbb{S}_\mu) \otimes \mathrm{id}$ acting on the the entangled states of the compound system $S + S$. In particular, $P_\mu(t)$ must correspond to a density matrix at all times $t \geq 0$.

Proposition 5.2 Set

$$R_4(t) := 1 + 2\mathrm{e}^{-2at}\frac{b}{\Omega}\sin(2\Omega t) \ , \tag{42}$$

the eigenvalues of $P_\mu(t)$ are positive if and only if

$$0 \leq \mu \leq \frac{1}{R_4} \ , \quad \text{where} \quad R_4 := 1 + 2\mathrm{e}^{-2at_*}\frac{b}{\sqrt{\Omega^2 + a^2}} \tag{43}$$

is the maximum of $R_4(t)$ which is achieved at $t_* = \frac{1}{2\Omega}\arcsin\frac{\Omega}{\sqrt{\Omega^2+a^2}}$.

Proof: Since the eigenvalues $e^\mu_{3,4}(t)$ interchange when $\sin(2\Omega t)$ changes sign, one can always consider $0 \leq t \leq \frac{\pi}{4\Omega}$. Then, the positivity of $e^\mu_4(t)$ requires $\mu\, R_4(t) \leq 1$ for all $t \geq 0$. Vice versa, suppose $\mu \leq R_4^{-1}(t)$ for all $t \geq 0$ and denote by

$$R_1(t) := 1 + 2\mathrm{e}^{-2at}\sqrt{1 + \frac{b^2}{\Omega^2}\sin^2(2\Omega t)} \qquad (44)$$

the term multiplying μ in the largest eigenvalue $e^\mu_1(t)$. Since $0 \leq \mu \leq 1$ and $b > 0$, one estimates: $\mu\, R_1(t) \leq 2 + \mu\, R_4(t) \leq 3$, whence $1 \geq e^\mu_1(t) \geq e^\mu_2(t) \geq e^\mu_3(t) \geq e^\mu_4(t) \geq 0$. ∎

In figures 1–3, the red line shows the time behaviour of the largest eigenvalue $e^\mu_1(t)$ and the blue line the lowest eigenvalue $e^\mu_4(t)$, for given values of a and b, with $\mu = 1$, $\mu > R_4^{-1}$ and $\mu < R_4^{-1}$. Both functions are computed with rescaled parameters $a \to a/\omega$, $b \to b/\omega$ and $t \to \omega t$; in this way $\Omega^2 = 1 - b^2$ with $0 \leq a, b \leq 1$. The green line corresponding to height 1 is showed for the sake of comparison with $e^\mu_1(t)$.

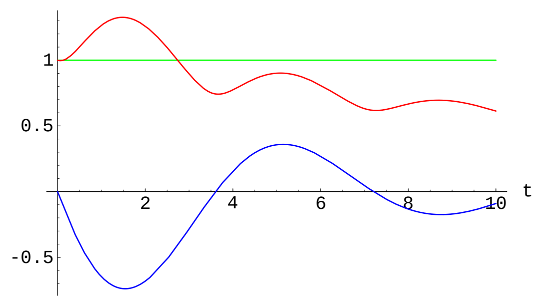

Figure 1: $a = 0.1$, $b = 0.9$, $R_4^{-1} = 0.25$, $\mu = 1$. Red line: $e^\mu_1(t)$. Blue line: $e^\mu_4(t)$

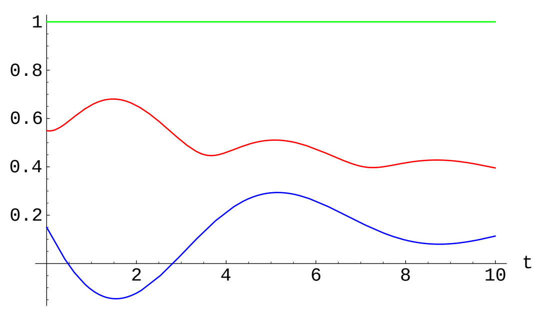

Figure 2: $a = 0.1$, $b = 0.9$, $R_4^{-1} = 0.25$, $\mu = 0.4$. Red line: $e^\mu_1(t)$. Blue line: $e^\mu_4(t)$

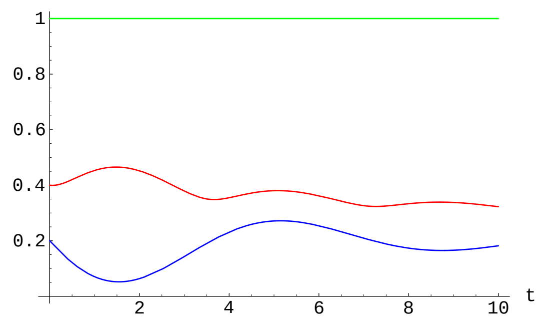

Figure 3: $a = 0.1$, $b = 0.9$, $R_4^{-1} = 0.25$, $\mu = 0.2$. Red line: $e_1^\mu(t)$. Blue line: $e_4^\mu(t)$

Remarks 5.2

1. The states P_μ that are obtained from the totally symmetric projector P by the action of the slippage operator $\mathbb{S}_\mu$ are the class of *isotropic states*, $P_\mu = \frac{1-\mu}{4} + \mu\, P$ in standard form. They are entangled if and only if $1 \geq \mu > \frac{1}{3}$ [18].

2. According to Remark 2.1.2, condition (43) makes $P_\mu(t) = (\gamma_t \circ \mathbb{S}_\mu) \otimes \mathrm{id}[P]$ a positive matrix for all $t \geq 0$, whence all slipped maps $\gamma_t \circ \mathbb{S}_\mu$ result completely positive, despite γ_t being possibly not even positive.

3. If $a^2 < b^2$, we have already seen that the channel $\mathbb{S}_\mu$ can cure non-positivity by multiplying the Pauli matrices by $\mu \leq R^{-1}$ where R is given in (28). From (27) and (42) it follows that $R(t) \leq R_4(t)$ so that $R \leq R_4$; namely, the maximal radius of the sphere of slipped initial conditions suggested by the single qubit non-positive dynamics is too large for keeping physical consistency against entanglement. The radius is to be decreased to at least R_4^{-1}.

In the following we shall see that even the value R_4^{-1} does not keep the non-positive dynamics free from pathologies; however, the ones we are going to expose are much more intriguing and subtler than the appearance of negative eigenvalues, being instead related to the creation of entanglement by means of the local action of maps of the form $\gamma_t \circ \mathbb{S}_\mu$.

We shall now consider $\mu \leq R_4^{-1}$ which ensures that the $P_\mu(t)$ are physical states of the compound system $S + S$ at all times $t \geq 0$. In order to study their entanglement content, we construct their concurrence $\mathcal{C}_\mu(t)$ (see (32)–(33)); one checks that $\widetilde{P}_\mu(t) = P_\mu(t)$, whence the square roots of the eigenvalues of $P_\mu(t)\widetilde{P}_\mu(t)$ are the values (38)–(41) and

$$\mathcal{C}_\mu(t) = \max\{0, c_\mu(t)\} \ , \qquad c_\mu(t) := \mu\, \mathrm{e}^{-2at}\sqrt{1+\frac{b^2}{\Omega^2}\sin^2(2\Omega t)} \ - \ \frac{1-\mu}{2} \ . \tag{45}$$

At $t = 0$, $c_\mu := \dfrac{3\mu - 1}{2} > 0$ if and only if $\mu > 1/3$, that is if and only if P_μ is entangled.

Proposition 5.3 The states $P_\mu(t)$ are entangled if and only if

$$\frac{1}{R_1(t)} < \mu \leq \frac{1}{R_4} \ . \tag{46}$$

Proof: The upper bound on μ guarantees that the $P_\mu(t)$ are well defined density matrices of the bipartite system $S+S$ at all $t \geq 0$, while the lower bound coming from (46), sets the range of $t \geq 0$ for which $\mathcal{C}_\mu(t) = c_\mu(t) > 0$. ■

Remark 5.3 Notice that by partially transposing $P_\mu(t)$ in (37), one exchanges B_t and C_t so that the eigenvalues of $T \otimes \mathrm{id}[P_\mu(t)]$, where T denotes transposition, are the same of those $P_\mu(t)$, but with $\mu \to -\mu$. Therefore, one checks that (46) corresponds to a non-positive partial transposed of $P_\mu(t)$, thus to an entangled $P_\mu(t)$, while $\mathcal{C}_\mu(t)$ quantifies the amount of entanglement it possesses.

We are now interested in the change of entanglement with time; from Proposition 5.1, we know $\mathcal{C}_\mu(t)$ cannot increase if γ_t is completely positive. Indeed, $\gamma_t \circ \mathbb{S}_\mu$, the composition of two completely positive maps, would also be completely positive and hence a physically consistent local operation.

Let us consider the time-derivative

$$\dot{c}_\mu(t) = \frac{2\mu\, \mathrm{e}^{-2at}}{\sqrt{1+\frac{b^2}{\Omega^2}\sin^2(2\Omega t)}}\, G(t) \ , \quad G(t) := \left[\frac{b^2\sqrt{\Omega^2+a^2}}{\Omega^2}\cos(2\Omega t + \varphi)\sin(2\Omega t) \ - \ a\right] \ , \tag{47}$$

where $\cos\varphi = \frac{\Omega}{\sqrt{\Omega^2+a^2}}$. One checks that the function within square brackets achieves its maximum

$$G := \max_{t \geq 0} G(t) = \frac{b^2}{2\Omega^2}\left(\sqrt{\Omega^2+a^2} \ - \ a\right) - a \ , \tag{48}$$

at $\overline{t} = t_*/2$.

Further, $G > 0$ if and only if $a^2 < \frac{b^4}{4\omega^2}$. This is only possible if $a^2 < b^2$ for $\omega^2 > b^2$, hence only if γ_t is non-positive; in such a case $c_\mu(t)$ increases in a neighborhood of $\bar{t}$ and does it independently of the value of μ. Therefore, if in the same neighborhood $\mathcal{C}_\mu(t) > 0$, then the entanglement of $P_\mu(t)$ increases by the local action of the non positive γ_t, an apparent physical inconsistency, for, as already noticed, physically, entanglement can be created by non-local operations only.

In order to check this possibility, we compare the bound $\mu\, R_4 \leq 1$, which is necessary to physical consistency of the $P_\mu(t)$ as states at all times, with the lower bound which ensures positive concurrence. Since we are interested in a neighborhood of $\bar{t}$, we shall consider $0 \leq \bar{t} + t$ as temporal parameter; then (46) gives

$$R_1(\bar{t}+t) > R_4 \Leftrightarrow f(t) := \mathrm{e}^{-2at}\sqrt{1+\frac{b^2}{\Omega^2}\sin^2(2\Omega(\bar{t}+t))} - \mathrm{e}^{-2a\bar{t}}\frac{b}{\sqrt{\Omega^2+a^2}} > 0\ . \quad (49)$$

In Figures 4–7, with the parameters a, b, ω rescaled as in figures 1–3, the red lines show where $f(t) > 0$, the blue ones where $g(t) := G(\bar{t}+t) > 0$ and thus the derivative in (47) is positive. It is then apparent that there are values of a, b with $a < b^2/(2\omega)$ ($a < b^2/2$ in the rescaled parameters) such that there exist time-intervals $t \in [t_1, t_2]$ where (49) holds together with $\dot{c}_\mu(\bar{t}+t) > 0$. For such choices of a and b, in order to avoid the unphysical creation of entanglement by means of local operations, one must exclude $\mu \in [R_1(\bar{t}+t)^{-1}, R_4^{-1}]$ with $t \in [t_1, t_2]$ and must thus enforce the stronger bound

$$0 \leq \mu \leq \frac{1}{\max_{t\in[t_1,t_2]} R_1(\bar{t}+t)}\ . \quad (50)$$

Furthermore, in the same figures 4–7, the green lines plot the function $r(t) := R_1(\bar{t}+t)-3$; then, figures 5 and 7 show that, in the interval $[t_1, t_2]$ where $f(t) > 0$ and $g(t) > 0$, by decreasing a, it also holds $r(t) \geq 0$, whence that $R_1(\bar{t}+t)^{-1} \leq 1/3$ and, consequently, also the upper bound in (50) is samller then $1/3$. According to Remark 5.2.1, it turns out that, in such cases, in order to ensure physical consistency, $\mathbb{S}_\mu$ must destroy all entangled isotropic states.

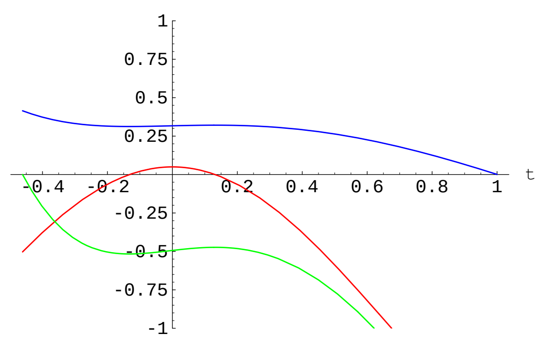

Figure 4: $a = 0.3$, $b = 0.8$. Red line: $f(t)$. Blue line: $g(t)$. Green line: $r(t)$

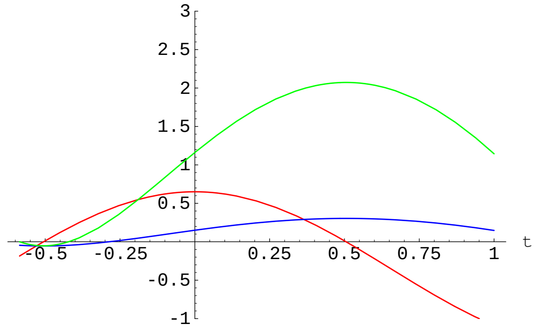

Figure 5: $a = 0.1$, $b = 0.8$. Red line: $f(t)$. Blue line: $g(t)$. Green line: $r(t)$

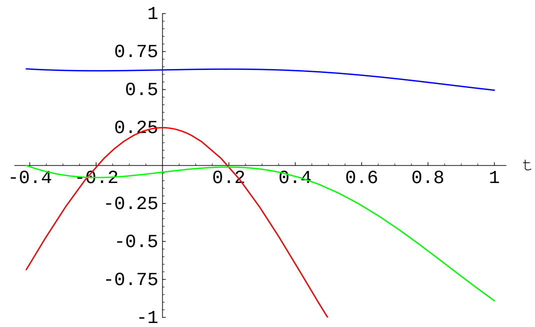

Figure 6: $a = 0.06$, $b = 0.4$. Red line: $f(t)$. Blue line: $g(t)$. Green line: $r(t)$

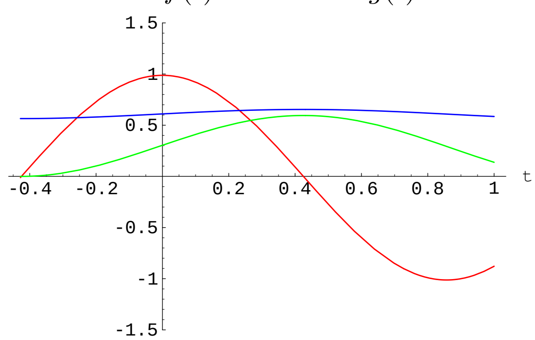

Figure 7: $a = 0.01$, $b = 0.4$. Red line: $f(t)$. Blue line: $g(t)$. Green line: $r(t)$

These results may appear surprising, but can be explained by looking at the structure of the states $P_\mu(t)$ and their entanglement characterization. Indeed, one knows that the so-called *entanglement of formation* [20] is a monotonically increasing function of the concurrence [21] and, moreover, that it cannot increase under local quantum operations. This seems to conflict with the fact that the local action of $\gamma_t \otimes \mathrm{id}$ may increase the concurrence and hence the entanglement of formation. However, there is no contradiction: in fact, γ_t is not positive, whereas a quantum operation is by definition completely positive. It is of some interest to inspect in more detail how non-positivity may generate entanglement by acting locally. Let us then consider the explicit expression of the entanglement of formation,

$$E(\rho) = \min_{\rho=\sum_i p_i\,\rho_i} \sum_i p_i\, S(\rho_i^1) \ , \tag{51}$$

where $S(\rho_i^1)$ is the von Neumann entropy of the marginal states of party 1 resulting from partial trace over party 2, $\rho_i^1 := \mathrm{Tr}_2(\rho_i)$, obtained from those that contribute to the convex decomposition $\rho = \sum_i p_i\rho_i$, $p_i \geq 0$, $\sum_i p_i = 1$.

Essentially, $E(\mathbb{G}\otimes\mathrm{id}[\rho]) \leq E(\rho)$ under a quantum operation $\mathbb{G}$ because any optimal decomposition $\sum_i p_i\rho_i$ of ρ achieving $E(\rho)$ provides a decomposition $\sum_i p_i\mathbb{G}\otimes\mathrm{id}[\rho_i]$ of $\mathbb{G}\otimes\mathrm{id}[\rho]$, whence the minimum can only decrease. In this section we have proved that there are times $0 < s < t$ such that $P_\mu(s) = \gamma_s\otimes\mathrm{id}[P_\mu]$ and $P_\mu(t) = \gamma_t\otimes\mathrm{id}[P_\mu]$ are well-defined entangled isotropic states satisfying $E(P_\mu(t)) > E(P_\mu(s))$. From the argument of above, such inequality can hold only if $\gamma_t\otimes\mathrm{id}$ acting on an optimal decomposition of $P_\mu(s)$ does not provide a decomposition of $P_\mu(t)$, namely only if at least one of the states optimally decomposing $P_\mu(s)$ are not mapped into density matrices by $\gamma_t\otimes\mathrm{id}$.

To summarize, by resorting to the slippage mechanism, one may force a non-positive time-evolution $\gamma_t\otimes\mathrm{id}$ to map a class of entangled states into states at all times, but it may happen that this local action increases their entanglement because it is the positivity of the spectrum of other entangled states, outside that class, which is spoiled in the course of time.

VI. CONCLUSION

We have considered a concrete model of reduced dynamics for a single qubit weakly interacting with a stochastic environment and derived a reduced dynamics depending on two parameters. The resulting semigroup consists of maps γ_t that range from non-positive to positive and completely positive.

The physical meaning of complete positivity is related to the existence of entangled states and can thus be fully appreciated only when the open qubit is correlated to an ancilla. Far from being abstract and out of experimental control, an entangled qubit pair one of whose parties experiences a noisy channel is a typical theoretical and practical context of quantum information and communication theory. In these cases, complete positivity cannot be dispensed with, otherwise physical inconsistencies immediately appear, typically the presence of negative probabilities in the spectrum of evolving entangled density matrices which lose their meaning as physical states.

In relation to entanglement, we have studied an approach, the so-called slippage of initial conditions, whereby complete positivity is deemed an abstract request devoid of physical content and as such refused as an unnecessary constraint on the open quantum dynamics. This means that one prefers to stick to a non-positive Markovian reduced dynamics and try to cure in some way or the other its pathological behavior. Indeed, already for a single qubit, one has to enforce a selection of the admissible initial states which is supposedly due to the action of the transient regime.

By means of a simple (completely positive) slippage operator $\mathbb{S}_\mu$, $0 \leq \mu \leq 1$, we have turned non-positive reduced dynamical maps γ_t into completely positive ones, $\gamma_t \circ \mathbb{S}_\mu$, that eliminate the presence of negative eigenvalues from the spectrum of time-evolving single qubit states.

However, we have showed that $\gamma_t \otimes \mathrm{id}$ acting on the isotropic states $P_\mu = \mathbb{S}_\mu \otimes \mathrm{id}[P]$, may increase their entanglement by acting locally. Such an unphysical possibility is due to the non-positivity of γ_t and can be eliminated by resorting to further slippage operators that may result in the whole elimination of entangled isotropic states.

Despite the simplicity of the model, we believe it contains the salient features of a more general structure: sticking to non-completely positive reduced dynamics, though cured by some suitable slippage mechanism, would conflict with the presence of entanglement. In some particular cases discussed in this paper, the conflict can only be avoided by the drastic elimination, via a suitable slippage, of whole classes of entangled states.

To conclude, slipped non-(completely) positive semigroups present two different contradictory aspects in relation to entanglement: on one hand, the slippage operated by the bath during the transient phase seems to hamper the possibility of creating entanglement by embedding two dynamically uncorrelated parties within a same environment, as discussed in [5]. On the other hand, unless all entangled states are eliminated by the slippage beforehand, the subsequent Markovian time-evolution may look like being able to create entanglement by acting locally. However, this entanglement creation is a spurious artifact: it is due to some entangled state, not destroyed by the slippage, which develops negative probabilities while evolving in time. As a consequence, curing non-positive Markovian evolutions through the slippage operation appears to be rather unsatisfactory and physically unviable.

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[22] The derivation of the reduced dynamics as sketched in Section II does not lead to either completely positive or positive γ_t, in general. In order to get a semigroup of such maps, a sufficient prescription is, roughly speaking, to formally integrate (2) and then operate an ergodic average of the kernel (see [2] for more details)